TOPOLOGICAL CHANGE IN MEAN CONVEX MEAN CURVATURE FLOW

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Abstract. Consider the mean curvature flow of an $(n+1)$-dimensional compact, mean convex region in Euclidean space (or, if $n < 7$, in a Riemannian manifold). We prove that elements of the m^{th} homotopy group of the complementary region can die only if there is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity for some $k \le m$. We also prove that for each m with $1 \le m \le n$, there is a nonempty open set of compact, mean convex regions K in $\mathbf{R}^{n+1}$ with smooth boundary ∂K for which the resulting mean curvature flow has a shrinking $\mathbf{S}^m \times \mathbf{R}^{n-m}$ singularity.

1. Introduction

Let $K(t)$ be a compact, time-dependent region in a Riemannian manifold such that the boundary $\partial K(t)$ moves by mean curvature flow. Clearly the topology of the complement $K(t)^c$ can change only if there is a singularity of the flow. It is natural to ask if we can deduce properties of the singularities from the way in which the topology changes. In this paper, we give a rather precise answer if the regions are mean convex. In particular, consider a mean curvature flow $t \in [0, \infty) \mapsto K(t)$ of compact regions in an $(n+1)$-dimensional Riemannian manifold N such that $K(0)$ is mean convex and has smooth boundary. If $n \ge 7$, we require that N be $\mathbf{R}^{n+1}$ with the Euclidean metric.[1]. We prove a theorem that implies the following:

1.1. **Theorem.** *Suppose that $0 \le a < b$ and that there is a map of the m-sphere into $K(a)^c$ that is homotopically trivial in $K(b)^c$ but not in $K(a)^c$.*

Then at some time t with $a \le t < b$, there is a singularity of the flow at which the Gaussian density is $\ge d_m$, the Gaussian density of a shrinking m-sphere in $\mathbf{R}^{m+1}$, and at which the tangent flow is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some k with $1 \le k \le m$.

The following is an interesting special case:

1.2. **Corollary.** *Suppose that K is a compact, mean convex subset of $\mathbf{R}^{n+1}$ with smooth boundary, and suppose that there is a map of the m-sphere into K^c that is homotopically nontrivial. Then the resulting mean curvature flow has a singularity at which the Gaussian density $\ge d_m$, and at which the tangent flow is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some k with $1 \le k \le m$.*

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[1]None of the arguments in this paper depend on dimension. However, they do require that the singularities of the flow have convex type (as defined in §2), and in high dimensions it has not been proved that all singularities have convex type except when the ambient space is Euclidean.

The corollary follows from the theorem because compact subsets of Euclidean space disappear in finite time under mean curvature flow. (Thus we can choose b large enough that $K(b)$ is empty.)

More generally, the topological assumption in Theorem 1.1 can be replaced by the weaker assumption that there is a continuous map

$$F : \mathbf{B}^{m+1} \to K(b)^c \text{ with } F(\partial \mathbf{B}^{m+1}) \subset K(a)^c$$

that cannot be homotoped by a 1-parameter family of such maps to a map F' whose image $F'(\mathbf{B}^{m+1})$ lies in $K(a)^c$. See Theorem 3.1. (The m in Theorem 3.1 corresponds to $(m+1)$ here.)

In Theorems 1.1 and 3.1, the moving hypersurfaces $\partial K(t)$ have no boundary. Those theorems generalize to hypersurfaces with boundary, where the motion of the boundary is prescribed: see Theorem 5.3. The theorems also generalize to flows in which the surfaces move with normal velocity equal to the mean curvature plus the normal component of a smooth vectorfield on the ambient space. See the discussion in §4.

The Gaussian density inequalities in the various theorems are sharp. For example, to see that the inequality in Corollary 1.2 is sharp in the case $n = 3$ and $m = 1$, let K be a thin, rotationally symmetric, solid torus in $\mathbf{R}^3$, i.e., the set of points at distance $\le \epsilon$ from a round circle in $\mathbf{R}^3$. If ϵ is sufficiently small, K will be mean convex and its complement will have nontrivial fundamental group. Furthermore, using the rotational symmetry, it is not hard to show that (under the flow) the surface collapses to a round circle, that each singularity is a shrinking $\mathbf{S}^1 \times \mathbf{R}^1$, and therefore that the Gaussian density is d_1. To see the sharpness for arbitrary dimensions n and m, let K be the set of points in $\mathbf{R}^{n+1}$ at distance $\le \epsilon$ from a round $(n-m)$-sphere in $\mathbf{R}^{n+1}$. Then K^c has nontrivial m^{th} homotopy and the singularities all have Gaussian density d_m.

The reader may wonder whether the density inequalities in the various theorems could be replaced by suitable equalities. For example, could Theorem 1.1 be strengthened to say that there is a singularity of Gaussian density d_k with k equal to the smallest m for which the theorem's hypotheses hold? The answer is no: even in the case of a single spacetime singularity, the change in topology does not determine the singularity type. For example, Altschuler, Angenent, and Giga [AAG95] proved that that there is a "doubly-degenerate neckpinch" mean curvature flow in $\mathbf{R}^3$: the moving surface is a smooth, axially symmetric, mean convex topological sphere until it collapses to a point, at which point the tangent flow is not a shrinking $\mathbf{S}^2$ but rather a shrinking $\mathbf{S}^1 \times \mathbf{R}$. Compare this flow to a flow consisting of shrinking round sphere in $\mathbf{R}^3$. The two flows are completely equivalent topologically, yet the singularities are different. In general, for every k and n with $1 \le k <$n, one can construct a mean convex n-sphere in $\mathbf{R}^{n+1}$ that shrinks to a point but whose tangent flow at that point is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$.

The theorems mentioned so far are about changes in topology of the complements $K(t)^c$ of the moving regions $K(t)$. One can also ask about changes in topology of the regions themselves. By standard topological duality theorems, the results described above imply results about the regions $K(t)$. See Section 6.

In Section 7, the results described above, in particular Theorem 1.1 and Corollary 1.2, are used to prove a result about persistence of singularities under perturbations of the initial surface: we prove that for $1 \le m \le n$, there is a nonempty open set of compact, mean convex regions in $\mathbf{R}^{n+1}$ with smooth boundary such

that the resulting mean curvature flow has a shrinking $\mathbf{S}^m \times \mathbf{R}^{n-m}$ singularity. To put this in context, let $\mathcal{C}_n$ be the collection of tangent flows at singularities of mean curvature flows of hypersurfaces in $\mathbf{R}^{n+1}$, where two tangent flows are identified if they are related by an ambient isometry. We can think of elements of $\mathcal{C}_n$ as singularity types. Let us call a singularity type T in $\mathcal{C}_n$ *avoidable* if the following holds: for a generic smooth, compact, emdedded hypersurface in $\mathbf{R}^{n+1}$, T does not occur as a tangent flow in the resulting mean curvature flow[2]. Otherwise, we call T *non-avoidable*. Thought experimentation suggests that the non-avoidable singularity types are precisely the shrinking spheres and cylinders, i.e., the $\mathbf{S}^m \times \mathbf{R}^{n-m}$. Theorem 7.1 shows that each $\mathbf{S}^m \times \mathbf{R}^{n-m}$ is indeed non-avoidable. In the other direction, Colding and Minicozzi [CM09] have made important progress toward proving that all other singularity types are avoidable. The most basic open question is whether n-planes of multiplicity > 1 are avoidable. (It seems likely that an n-plane of multiplicity > 1 is not merely avoidable, but that in fact it can never occur as a tangent flow if the initial surface is compact and smoothly embedded.)

The theorems in this paper all assume mean convexity. The reader may wonder what happens if that assumption is dropped. Certain analogs of the theorems hold for arbitrary (i.e., not necessarily mean convex) hypersurfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$; see [IW11c]. For hypersurfaces (of any dimension) that do not fatten under level set flow, there are some restrictions on the way that the topology of the complement can change, no matter what kinds of singularities occur. See [Whi95].

In [IW11c], the results of this paper are used to get lower Gaussian density bounds on self-similar shrinkers for mean curvature flow, and in [IW11a] and [IW11b] they are used to get lower bounds for densities of minimal cones.

The results of this paper rely strongly on properties of singularities of mean convex mean curvature flow that were proved in [Whi00] and [Whi03]. However, the density bounds in this paper are vacuously true for flows that have any singularities with Gaussian density ≥ 2. Thus for the density bounds in the paper, one only needs the results of [Whi00] and [Whi03] under the assumption that the singularities have Gaussian density < 2, and the most complicated parts of those papers are trivially true under that assumption.

2. Preliminaries

In this section, we state the facts about mean curvature flow of mean convex sets that are important for this paper. Let $t \in [0, \infty) \mapsto K(t)$ be a mean curvature flow of mean convex subsets of a smooth, $(n+1)$-dimensional Riemannian manifold. If $x \in \partial K(t)$ is a regular point, we let $\kappa_1(x) \leq \kappa_2(x) \leq \cdots \leq \kappa_n(x)$ be the principal curvatures of $\partial K(t)$ with respect to the inward unit normal normal. (We could also write $\kappa_i(x,t)$, but since the surfaces $\partial K(t)$ for distinct values of t are disjoint, t is determined by x.) We let $h(x) = \kappa_1(x) + \cdots + \kappa_n(x) > 0$ be the scalar mean curvature. We say that a singular point $x \in \partial K(t)$ (where $t > 0$) has *convex type* provided

(1) Each tangent flow at x is a self-similarly shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k \geq 1$.

[2] In general, one does not have uniqueness (after the first singularity) for mean curvature flow: there may be more than one mean curvature flow with a given initial surface. However, in the definition of avoidable singularity types, only generic initial surfaces matter. For generic initial surfaces, the flow is unique and so one may speak of *the* resulting mean curvature flow.

(2) If $x_i \in \partial K(t_i)$ is a sequence of regular points converging to x, then

$$\liminf \frac{\kappa_1(x_i)}{h(x_i)} \geq 0.$$

(Actually (1) follows from (2), which in turn follows from the seemingly weaker assumption that the lim inf in (2) is $> -\infty$ for every such sequence x_i. See [Whi03].)

In many situations, singularities are known to have convex type:

2.1. **Proposition.** *Suppose that K is a compact, mean convex region in an $(n+1)$-dimensional Riemannian manifold. Let $t \in [0,\infty) \mapsto K(t)$ be the mean curvature flow with $K(0) = K$. Suppose that*

(1) $n < 7$, *or*
(2) ∂K *is smooth and* $N = \mathbf{R}^{n+1}$.

Then for $t > 0$, the singularities of the flow all have convex type.

See [Whi03] for the proof in the case (1) and [Whi11] for the proof in case (2).

2.2. **Proposition.** *Let $t \in [a,b] \mapsto K(t)$ be a mean curvature flow of mean convex sets, and suppose that the singularities of the flow have convex type. Let $t \in (a,b]$ and let x be a point in the interior of $K(t)$. Let y be a point in $\partial K(t)$ that minimizes distance to x. Then y is a regular point of the flow.*

Proof. Note that $K(t)$ contains the ball with center x and radius $\mathrm{dist}(x,y)$, from which it follows

$$\liminf_{r\to 0} \frac{\mathrm{vol}(K(t)\cap \mathbf{B}(y,r))}{\mathrm{vol}(\mathbf{B}(y,r))} \geq \frac{1}{2}.$$

On the other hand, if $z \in \partial K(t)$ is a singular point of convex type, then it is straightforward to show that

$$\lim_{r\to 0} \frac{\mathrm{vol}(K(t)\cap \mathbf{B}(z,r))}{\mathrm{vol}(\mathbf{B}(z,r))} = 0.$$

□

2.3. **Proposition** (Stone)**.** *Let x be a convex-type singularity of a mean convex mean curvature flow. Then there is a $k = k(x) \geq 1$ such that every tangent flow at x is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$, where k depends only on the Gaussian density Θ at the point x. (It does not depend on the sequence of spacetime dilations used to obtain the tangent flow.)*

Thus the tangent flow is unique up to rotations. For the reader's convenience, we give the idea of Stone's proof. See [Sto94, Appendix A] for details.

Proof. The Gaussian density d_k of a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ (where $\mathbf{S}^k$ is the unit k-sphere in $\mathbf{R}^{k+1}$) may be calculated explicitly:

$$\begin{aligned} d_k &= \left(\frac{k}{2\pi e}\right)^{k/2} \sigma_k \\ &= \left(\frac{k}{2e}\right)^{k/2} \left(\frac{2\sqrt{\pi}}{\Gamma(\frac{k+1}{2})}\right), \end{aligned}$$

where σ_k is the area of a k-dimensional sphere of radius 1. (As the notation indicates, the value of d_k turns out not to depend on n.) Using this formula, one can show that

$$d_1 > d_2 > \dots . \tag{1}$$

Now if a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ is a tangent flow to $t \mapsto K(t)$ at the point (x,t), then $d_k = \Theta$. Thus by (1), k is determined by Θ. □

2.4. **Proposition.** *Let Σ_k be the set of spacetime points (x,t) such that*

(1) $t > 0$,
(2) $x \in \partial K(t)$,
(3) x is a singular point of convex type, and
(4) the Gaussian density at (x,t) is d_k (or, equivalently, the tangent flows at x are shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$s.)

Then Σ_k has parabolic Hausdorff dimension at most $(n-k)$.

This follows easily from standard dimension reducing. (It also a special case of the stratification theory in [Whi97, §9].) Actually, in this paper, we do not need the full strength of Proposition 2.4. All we need is the following much weaker corollary:

2.5. **Corollary.** *Suppose that at a certain time $t > 0$, the singularities all have convex type with Gaussian density $\le d_k$. Then $K(t)$ is a smooth $(n+1)$-manifold with boundary except for a closed subset of $\partial K(t)$ whose Hausdorff dimension is at most $(n-k)$.*

2.6. **Proposition.** *Let $t > 0$ and let $p \in \partial K(t)$ be a either a regular point or a convex-type singular point at which the Gaussian density $\Theta(p)$ of the flow is $\le d_m$. Let x_i be a sequence of points in the interior of $K(t)$ that converge to p. Let y_i be a point in $\partial K(t)$ that minimizes distance to x_i. Translate $K(t)$ by $-y_i$ and dilate by $1/\operatorname{dist}(x_i,y_i)$ to get a set K_i. Then a subsequence $K_{i(j)}$ converges to a convex set K' with smooth boundary, and the convergence $\partial K_{i(j)} \to \partial K'$ is smooth on bounded sets.*

Furthermore, the homotopy groups $\pi_j(\partial K')$ are trivial for $j < m$.

Proof. The assertion is trivially true if p is a regular point (in that case, the set K' is a closed halfspace), so we assume that p is a singular point. Except for the assertion about homotopy groups, this is proved in [Whi03], which also shows that, after a rotation, either

(i) ∂K is the graph of an entire function from $\mathbf{R}^n$ to $\mathbf{R}$, or
(ii) K has the form $C \times \mathbf{R}^{n-k}$ for some $k \ge 1$, where C is a compact, convex subset of $\mathbf{R}^{k+1}$.

In the first case, all the homotopy groups of ∂K are trivial. Thus we may assume that K has the form $C \times \mathbf{R}^{n-k}$ as in (ii).

Recall that the *entropy* of a hypersurface M in $\mathbf{R}^{n+1}$ is the supremum of

$$\frac{1}{(4\pi)^{n/2} r^n} \int_{y \in M} e^{-|y-x|^2/4r^2} \, d\mathcal{H}^n y$$

over all $x \in \mathbf{R}^{n+1}$ and $r > 0$. Because ∂K is a part of a limit flow at (p,t), its entropy is at most the Gaussian density of the original flow at the point p:

$$\operatorname{Entropy}(\partial K) \le \Theta(p) \le d_m. \tag{2}$$

(This follows easily from Huisken's monotonicity.) On the other hand, ∂K forms an $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity under mean curvature flow. (If this not clear, apply Huisken's Theorem [Hui84] to see that the mean curvature flow starting with C collapses to a round point, and then cross that flow with $\mathbf{R}^{n-k}$ to get a mean curvature flow starting with K and collapsing to a $(n-k)$-space with an $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity.) By Huisken's monotonicity,

$$d_k \leq \text{Entropy}(\partial K),$$

so $d_k \leq d_m$ by (2). Thus $k \geq m$ (by (1)), which implies that the j^{th} homotopy group of ∂K (which is diffeomorphic to $\mathbf{S}^k \times \mathbf{R}^{n-k}$) is trivial for all $j < m$. □

3. The Main Theorem

We begin by recalling some topological terminology. Suppose that Y is a topological space and that X is a subset of Y. We write

$$F : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

to indicate that F is a continuous map of the pair $(\mathbf{B}^k, \partial \mathbf{B}^k)$ into (Y, X), i.e, a continuous map of $\mathbf{B}^k$ into Y such that $F(\partial \mathbf{B}^k) \subset X$. Two such maps

$$F, G : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

are called homotopic in (Y, X) provided there is a homotopy $H : \mathbf{B}^k \times [0, 1] \to Y$ from F to G such that

$$H(\cdot, s) : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

for all $s \in [0, 1]$.

We say that the pair (Y, X) is **m-connected** if for every $k \leq m$, every continuous map

$$F : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

is homotopic in (Y, X) to a map G whose image $G(\mathbf{B}^k)$ lies in X.

We can now state the main theorem:

3.1. **Theorem.** *Let $t \in [0, \infty) \mapsto K(t)$ be a mean curvature flow of compact, mean convex subsets of a Riemannian manifold N. Suppose that $0 < a < b$ and that each singularity during the the time interval $a \leq t < b$ has convex type and has Gaussian density $\leq d_m$, the Gaussian density of a shrinking m-sphere in $\mathbf{R}^{m+1}$.*

Then the pair $(K(b)^c, K(a)^c)$ is m-connected.

In particular, the conclusion implies that if a map of $\mathbf{S}^k$ (for $k < m$) into $K(a)^c$ is contractible in $K(b)^c$, then it is also contractible in $K(a)^c$. Thus Theorem 3.1 implies Theorem 1.1.

Recall (Proposition 2.1) that if $N = \mathbf{R}^{n+1}$ and $\partial K(0)$ is smooth, or if $n < 7$, then the singularities all have convex type.

Note also that if $\partial K(0)$ is smooth, then we can also allow $a = 0$ in Theorem 3.1, because the topology cannot not change before the first singular time.

Proof of Theorem 3.1. The theorem follows immediately from Theorem 4.4 and Proposition 4.3 below. □

4. An Abstract Form of the Main Theorem

In this section, we state and prove an abstract version (Theorem 4.4) of Theorem 3.1. The abstract version is no harder to prove than the special case, but it has the advantage of also applying to some variants of mean curvature flow. For example, for $n < 7$, it also applies to regions $t \mapsto K(t)$ in $\mathbf{R}^{n+1}$ (or in an $(n+1)$-dimensional Riemannian manifold) whose boundaries evolve with velocity given by

$$\mathbf{v} = H + F^{\perp} \tag{3}$$

where the drift F is a smooth, time-independent vectorfield on the ambient space. The relevant analog of mean convexity is the condition that at time 0, the velocity $\mathbf{v}$ is everywhere nonzero and points into the region $K(0)$.

(Existence of a set-theoretic solution of (3) may be proved as in [Ilm93] or [Whi95]. Existence of an associated flow of varifolds may be proved by the methods of Evans-Spruck [ES91], which are adapted to general ambient manifolds in [Ilm92]. In particular, as in [Ilm92, §5], one gets the set-theoretic flow by taking a limit of of flows of moving graphs in $N \times \mathbf{R}$ and then slicing to get the flowing sets in N. Taking the corresponding varifold limits gives the appropriate varifold flow in $N \times \mathbf{R}$, from which one gets the varifold flow in N by slicing. Once one knows (or assumes) existence of the set-theoretic flow and of the associated flow of varifolds, the partial regularity proofs in [Whi00] carry over with only minor and straightforward modifications, as does the proof in [Whi03] that the singularities have convex type when $n < 7$. The proof in [Whi11] of convex-type in $\mathbf{R}^{n+1}$ for $n \geq 7$ is more delicate and does not seem to generalize to flows (3) with a drift term.)

4.1. **Definition.** Suppose K is a closed set in the interior[3] of a smooth $(n+1)$-dimensional Riemannian manifold. A point in K is a **regular** point of K if it is an interior point of K or if it is a boundary point with a neighborhood U such that $K \cap U$ is smoothly diffeomorphic to a closed halfspace in $\mathbf{R}^{n+1}$. The **singular set** $\operatorname{sing}(K)$ of K is the set of points in K that are not regular points of K.

Note that the singular set of K is a closed subset of ∂K.

The quantity $Q(K)$ introduced in the following definition may at first seem peculiar, but for mean convex mean curvature flow, the condition $Q(K(t)) \geq m$ is closely related to the condition "the singularities at t have Gaussian densities $\leq d_m$". See Proposition 4.3 and the discussion following.

4.2. **Definition.** Suppose K is a closed set in the interior of a smooth $(n+1)$-dimensional Riemannian manifold. We define $Q(K)$ to be the largest integer m with the following properties:

(a) The singular set $\operatorname{sing}(K)$ has Hausdorff dimension $\leq n - m$.
(b) Let x_i be a sequence of points in the interior of K converging to a point in ∂K. Translate K by $-x_i$ and dilate by $1/\operatorname{dist}(x_i, \partial K)$ to get K_i. Then a subsequence of the K_i converges to a convex subset K' of $\mathbf{R}^{n+1}$ with smooth boundary, and the convergence is smooth on bounded sets.
(c) If K' is as in (b), then $\partial K'$ has trivial k^{th} homotopy for every $k < m$.

If no such integer exists, we let $Q(K) = -\infty$.

[3]For now, the reader may as well assume that the ambient manifold has no boundary. In §5, we will consider sets K that contain a portion of the boundary of the ambient manifold.

Note that (b) implies

$$\text{If } x \in \text{interior}(K) \text{ and if } y \text{ is point in } \partial K \text{ closest to } x, \\ \text{then } y \text{ is a regular point of } K. \tag{4}$$

(If this is not clear, consider a sequence of points x_i lying on the geodesic between x and y and converging to y. The limit set K' in (b) must be a halfspace, and the smooth convergence in (b) then implies that y is a regular point.)

Note also that if K has no interior, then (b) and (c) are vacuously true, and $\text{sing}(K) = K$, so in that case $Q(K)$ is the largest integer less than or equal to $n - \dim(\text{sing}(K))$.

The following proposition describes for mean curvature flow how $Q(K(t))$ is related to the Gaussian densities of the singularities at time t:

4.3. **Proposition.** *Let $t \in [0,T] \mapsto K(t)$ be a mean curvature flow of mean-convex regions in the interior of a smooth Riemannian $(n+1)$-manifold. If $t \in (0,T]$ and if the singularities at time t all have convex type with Gaussian densities $\le d_m$, then $Q(K(t)) \ge m$.*

Proof. The result follows immediately from Proposition 2.2, Corollary 2.5, and Proposition 2.6. □

For mean convex mean curvature flow, $Q(K(t))$ will typically equal the smallest m such that there is a singularity at time t with Gaussian density d_m. However, there are degenerate situations in which $Q(K(t))$ is strictly less than that m. For example, at the singular time for the doubly-degenerate neckpinch in $\mathbf{R}^3$ mentioned in the introduction, $K(t)$ is a single point and thus $Q(K(t)) = 2 - 0 = 2$, but the Gaussian density at that singularity is d_1, not d_2.

4.4. **Theorem.** *Let $t \in [a,b] \mapsto K(t)$ be a one-parameter family of compact sets in the interior of a smooth, Riemannian $(n+1)$-manifold such that*

$$K(t) \subset K(T) \textit{ for } a \le T \le t < b$$

and such that the boundaries $M(t) := \partial K(t)$ form a partition of $K(a)\backslash\text{interior}(K(b))$. Assume that

$$Q(K(T)) \ge m \textit{ for each } T \in [a,b). \tag{5}$$

Then the pair $(K(b)^c, K(a)^c)$ is m-connected.

4.5. **Remark.** Note that the hypotheses in the first sentence of the theorem imply that there is a continuous function $\tau : K(a) \to \mathbf{R}$ such that

$$M(t) = \{x : \tau(x) = t\} \text{ for } t \in [a,b), \text{ and} \\ K(t) = \{x : \tau(x) \ge t\} \text{ for } t \in [a,b].$$

If $\tau(x) \in [a,b]$, then $\tau(x)$ is the time at which the moving surface $\partial K(t)$ passes through the point x.

Proof. Let $k \le m$ and let

$$F_0 : (\mathbf{B}^k, \partial\mathbf{B}^k) \to (K(b)^c, K(a)^c).$$

be a continuous map. Let $\mathcal{F}$ be the set of all continuous maps

$$F : (\mathbf{B}^k, \partial\mathbf{B}^k) \to (K(b)^c, K(a)^c)$$

such that F is homotopic in $(K(b)^c, K(a)^c)$ to F_0. We must show that $\mathcal{F}$ contains a map whose image lies in $K(a)^c$, i.e., a map whose image is disjoint from $K(a)$.

Equivalently, if J is the set of $t \in [a,b]$ such that $\mathcal{F}$ contains a map F whose image is disjoint from $K(t)$, then we must show that $a \in J$.

We will prove that $J = [a,b]$ (and therefore that $a \in J$) by proving the following four statements:

(i) $b \in J$.
(ii) J is a relatively open subinterval of $[a,b]$.
(iii) If T is in the closure of J, then $\mathcal{F}$ contains a map F whose image is contained in the union of $K(T)^c$ and the regular part of $\partial K(T)$.
(iv) If T is in the closure of J, then $T \in J$.

Statements (i), (ii), and (iv) imply that J is a nonempty subinterval of $[a,b]$ that is both open and closed in $[a,b]$, and therefore that J is all of $[a,b]$, as desired.

Statement (i) is trivially true (since $F_0 \in \mathcal{F}$ and $F_0(\mathbf{B}^k)$ is disjoint from $K(b)$.)

Next we prove statement (ii). For $F \in \mathcal{F}$, let

$$J_F := \{t \in [a,b] : F(\mathbf{B}^k) \cap K(t) = \emptyset\}.$$

Note that

$$J = \cup_{F\in\mathcal{F}} J_F. \tag{6}$$

By definition of $\mathcal{F}$, the set J_F contains b. Since the $K(t)$'s are nested, if $a \le t \le t' \le b$ and if t is in J_F, then t' is also in J_F. Thus J_F is an interval containing b. We claim that J_F is relatively open in $[a,b]$. If $a \in J_F$, then $J_F = [a,b]$, which is certainly relatively open in $[a,b]$. Thus suppose $a \notin J$, i.e., that $K(a)$ intersects $F(\mathbf{B}^k)$. Note that there is a last time t such that $K(t)$ intersects $F(\mathbf{B}^k)$. (Indeed, $t = \max\{\tau(x) : x \in F(\mathbf{B}^k)\}$, where $\tau(\cdot)$ is the function in Remark 4.5.) Then $J_F = (t,b]$, which is relatively open in $[a,b]$. We have shown that each J_F is a relatively open subinterval of $[a,b]$ containing b. Hence their union J is also such a subinterval of $[a,b]$. This proves statement (ii).

Next we observe that statement (iii) implies statement (iv). For suppose T is in the closure of J. Then, assuming that statement (iii) holds, $\mathcal{F}$ contains a map F that lies in the union of $K(T)^c$ with the regular part of $\partial K(T)$. Now we simply push $F(\mathbf{B}^k)$ into $K(T)^c$ by pushing it (where it touches the regular part of $\partial K(T)$) in the direction of the outward unit normal to $K(T)$. Thus $T \in J$, which completes the proof that statement (iii) implies statement (iv).

(The sentence "now we simply push…" may be made more precise as follows. Let $S = F(\mathbf{B}^k) \cap \partial K(T)$. Let $\mathbf{v}$ be a smooth, compactly supported vectorfield defined on the regular part of $\partial K(T)$ such that $\mathbf{v}$ is nonzero at every point of S and such that at each point, $\mathbf{v}$ is a nonnegative multiple of the outward unit normal to $\partial K(T)$. Now extend $\mathbf{v}$ to be a smooth vectorfield on the ambient space that vanishes outside of $K(a)$. The flow generated by $\mathbf{v}$ homotopes F to a map in $\mathcal{F}$ whose image is disjoint from $K(T)$.)

It remains only to show statement (iii). Suppose $T \in [a,b)$ is in the closure of J. Let $\epsilon > 0$ (to be specified later). By statements (i) and (ii), there exist $T^* \in J \cap (T,b]$ arbitrarily close to T. Choose such a T^* sufficiently close to T that every point in $K(T) \setminus K(T^*)$ is within distance $< \epsilon$ of $\partial K(T)$. (This is possible by the continuity of the function $\tau(\cdot)$ in Remark 4.5.)

Since $T^* \in J$, there is a map $F \in \mathcal{F}$ such that $F(\mathbf{B}^k)$ is disjoint from $K(T^*)$. We may assume that F is smooth since the C^∞ maps are dense in the set of continuous

maps. Now

$$\dim(\operatorname{sing}(K(T)) \leq n - Q(K(T)) \leq n - m,$$

and therefore since $k \leq m$,

$$\dim(\operatorname{sing}(\partial K(T)) + k \leq n < n + 1.$$

Consequently, we may assume, by putting F in general position, that $F(\mathbf{B}^k)$ contains no singular points of $\partial K(T)$. (See the appendix if this is not clear.)

We will construct a map G from $\mathbf{B}^k$ such that the image of G is contained in $K(T)^c$ together with the regular part of $\partial K(T)$. We will also construct a homotopy from F to G in $(K(b)^c, K(a)^c)$. The homotopy shows that $G \in \mathcal{F}$, thus establishing statement (iii).

Let $\Omega \subset \mathbf{B}^k$ be the inverse image under F of the interior of $K(T)$. We now describe the construction of the map G on the open set Ω.

First some terminology. Recall that a d-simplex is the convex hull of $(d+1)$ points in a Euclidean space provided those $(d+1)$ points do not lie in any affine subspace of dimension $< d$. The points are called vertices of the simplex. If the distance between each pair of vertices is 1, we say that the simplex is **standard**. Note that any two d-simplices are affinely isomorphic. In particular, given any d-simplex Δ, there is an affine bijection $\sigma : \Delta \to \Delta_s$ from Δ to a standard simplex Δ_s. We define the **standardized distance** $d_s(\cdot, \cdot)$ on Δ by

$$d_s(x, y) = |\sigma(x) - \sigma(y)|.$$

Given a map F from Δ into a metric space Z, we define the **standardized Lipschitz constant** $\operatorname{Lip}_s(F)$ of F to be the Lipschitz constant of F with respect to the standardized distance on Δ:

$$\operatorname{Lip}_s(F) = \sup_{x \neq y} \frac{\operatorname{dist}(F(x), F(y))}{d_s(x, y)}.$$

We now describe the map G on the portion Ω of $\mathbf{B}^k$. (Later we will extend G to all of $\mathbf{B}^k$ by letting $G = F$ on $\mathbf{B}^k \setminus \Omega$.) First, triangulate Ω. By refining the triangulation, we may assume that for each simplex Δ of the triangulation,

$$\operatorname{diam}(F(\Delta)) < \epsilon \operatorname{dist}(F(\Delta), \partial K(T)).$$

Here $\operatorname{dist}(X, Y)$ denotes the infimum of $\operatorname{dist}(x, y)$ among all $x \in X$ and $y \in Y$.

We define G on Ω inductively by defining it first on the 0-skeleton of the triangulation of Ω, then on the 1-skeleton, and so on. For each vertex v in the 0-skeleton, we choose a point $q \in \partial K(T)$ that minimizes $\operatorname{dist}(q, F(v))$, and we then let $G(v)$ be that chosen q. Note that q is a regular point of $\partial K(T)$ by (4). Having defined G on the $(j-1)$-skeleton of Ω, we extend it to the j-skeleton as follows. For each j-simplex Δ in the triangulation, we choose a map

$$g : \Delta \to \partial K(T)$$

that minimizes $\operatorname{Lip}_s(g)$ among all maps $g : \Delta \to \partial K(T)$ such that $g = G$ on $\partial \Delta$. Having chosen such a g, we let $G(x) = g(x)$ for $x \in \Delta$. (In Lemma 4.6 below, any map G constructed by this inductive procedure will be called "F-optimal".)

Of course we must check that the procedure does not break down in going from the $(j-1)$-skeleton to the j-skeleton. That it does not break down is proved below in Lemma 4.6 (provided $\epsilon > 0$ is sufficiently small). The lemma shows (for all sufficiently small $\epsilon > 0$) that:

(7) $G(\Omega)$ lies in the regular part of $\partial K(T)$, and

(8) $\operatorname{dist}(F(x), G(x)) \le C \operatorname{dist}(F(x), \partial K(T))$ for all $x \in \Omega$. (See (9) in the lemma.)

By (8), the map G extends continuously to $\mathbf{B}^k$ by setting $G(x) = F(x)$ for $x \in \mathbf{B}^k \setminus \Omega$.

Now define a homotopy $H : \mathbf{B}^k \times [0,1] \to K$ from F to G by setting

$$H(x,s) = (1-s)F(x) + sG(x)$$

if the ambient space is Euclidean. More generally, we define H by letting $H(x,\cdot) : [0,1] \to K(T)$ be the unique shortest geodesic (parametrized with constant speed) joining $F(x)$ to $G(x)$. (By (8), the shortest geodesic will be unique if $\epsilon > 0$ is sufficiently small, since $\operatorname{dist}(F(x), \partial K(T)) < \epsilon$.)

It remains only to show that (if $\epsilon > 0$ is sufficiently small) the image of H is disjoint from $K(b)$, i.e., that for $x \in \Omega$, the geodesic from $F(x)$ to $G(x)$ is disjoint from $K(b)$. Choose ϵ with

$$0 < \epsilon < \frac{\operatorname{dist}(\partial K(T), K(b))}{C}.$$

(This is possible since $\partial K(T)$ and $K(b)$ are disjoint.) Thus by (8),

$$\operatorname{dist}(F(x), G(x)) < \operatorname{dist}(\partial K(T), K(b)).$$

This means that the geodesic from $G(x)$ (which is in $\partial K(T)$) to $F(x)$ is too short to reach $K(b)$. Thus that geodesic is disjoint from $K(b)$.

We have proved that the image of the homotopy H is disjoint from $K(b)$. The homotopy proves that $G \in \mathcal{F}$. This completes the proof of Theorem 4.4. □

We now turn to the lemma that was used in the proof of Theorem 4.4. First we need some terminology. Fix a $T > 0$ and let $K = K(T)$. Let X be a simplicial complex and let F be a map from X to K. We say that a map $G : X \to \partial K$ is **F-optimal** provided:

(1) For each vertex v of X, $G(v)$ realizes the minimum distance from a point in ∂K to $F(v)$:
$$\operatorname{dist}(F(v), \partial K) = \operatorname{dist}(F(v), G(v)).$$

(2) For each simplex Δ of X, the restriction $G|\Delta$ is a Lip_s-minimizing map from Δ to ∂K. That is, if $g : \Delta \to \partial K$ is any map such that $g|\partial\Delta = G|\partial\Delta$, then
$$\operatorname{Lip}_s(G|\Delta) \le \operatorname{Lip}_s(g).$$

4.6. **Lemma.** *Let K be a compact subset of the interior of a smooth, $(n+1)$-dimensional manifold. Let Δ be a simplex of dimension $k \le Q(K)$. Then there is an $\epsilon > 0$ and a $C < \infty$ with the following property. If $F : \Delta \to K$ is a map such that*

$$\operatorname{diam}(F(\Delta)) < \epsilon \operatorname{dist}(F(\Delta), \partial K)$$

and such that

$$\operatorname{dist}(F(\Delta), \partial K) < \epsilon,$$

then each F-optimal map from $\partial\Delta$ to ∂K extends to an F-optimal map G from Δ to ∂K, and for any such extension G,

$$\operatorname{Lip}_s(G) \le C \operatorname{dist}(F(\Delta), \partial K),$$

and

$$\operatorname{diam}(F(\Delta) \cup G(\Delta)) \le C \operatorname{dist}(F(\Delta), \partial K). \tag{9}$$

We may assume that the simplex Δ is standard since the statement of the theorem is not affected by affine reparametrizations of the domain. For purposes of proof, it is convenient to restate the lemma as follows:

4.7. **Lemma.** *Let K be as in Lemma 4.6, and let Δ be a standard simplex of dimension $k \le Q(K)$. Let $\epsilon_i \to 0$, and suppose that $F_i : \Delta \to K$ is a sequence of maps such that*

$$\operatorname{diam}(F_i(\Delta)) \le \epsilon_i \operatorname{dist}(F_i(\Delta), \partial K) \tag{10}$$

and such that

$$\operatorname{dist}(F_i(\Delta), \partial K) < \epsilon_i. \tag{11}$$

Suppose also that $\Gamma_i : \partial\Delta \to \partial K$ is a sequence of F_i-optimal maps. Then for all sufficiently large i, there exists an F_i-optimal map $G_i : \Delta \to \partial K$ that extends Γ_i, and such a G_i must (for all sufficiently large i) have the following properties:

(i) *$G_i(\Delta)$ is contained in the regular part of ∂K.*
(ii) *The quantities*
$$\frac{\operatorname{Lip} G_i}{\operatorname{dist}(F_i(\Delta), \partial K)}$$
(if $k > 0$) and
$$\frac{\operatorname{diam}(F_i(\Delta) \cup G_i(\Delta))}{\operatorname{dist}(F_i(\Delta), \partial K)}$$
are bounded above as $i \to \infty$.

Proof. We prove it by induction on the dimension of Δ.

If Δ is 0-dimensional, it is a single point p. Let $G_i(p)$ be a point in the interior of ∂K such that
$$\operatorname{dist}(F_i(p), G_i(p)) = \operatorname{dist}(F_i(p), \partial K).$$
Since $Q(K) > -\infty$, this implies that $G_i(p)$ is a regular point (see (4)), so (i) holds. The two ratios in (ii) are trivially equal to 0 and 1, so (ii) also holds. This completes the proof of the lemma when Δ is 0-dimensional.

Now suppose that $1 \le k = \dim(\Delta) \le Q(K)$. By induction, we may assume that the lemma is true for each face of Δ. Let p_i be a point in $F_i(\Delta)$ that minimizes the distance from $F_i(p_i)$ to ∂K.

Translate K by $-F(p_i)$ and dilate by
$$\lambda_i = \frac{1}{\operatorname{dist}(F_i(p), \partial K)}$$
to get a set K_i'. Let $F_i' : S \to K_i'$ and $\Gamma_i' : \partial\Delta \to \partial K_i'$ be the maps corresponding to F_i and Γ_i. Note that

$$0 \in F_i'(\Delta) \subset K_i' \tag{12}$$

and that

$$1 = \operatorname{dist}(0, \partial K') = \operatorname{dist}(F_i'(\Delta), \partial K_i'). \tag{13}$$

By passing to a subsequence, we may assume that the K_i' converge smoothly to a convex set K' with

$$0 \in K' \text{ and } \operatorname{dist}(0, \partial K') = 1. \tag{14}$$

By (10) and (13),
$$\operatorname{diam}(F_i'(\Delta)) \le \epsilon_i \operatorname{dist}(F_i'(\Delta), \partial K_i') = \epsilon_i \to 0,$$

so by (12),

$$F_i'(\cdot) \to 0 \text{ uniformly.} \tag{15}$$

Thus by (14),

$$\operatorname{dist}(F_i'(\cdot), \partial K_i') \to 1 \quad \text{uniformly.} \tag{16}$$

By induction we can assume that (ii) holds for the restrictions of F_i and Γ_i to each face Δ^* of Δ. Thus

$$\operatorname{Lip}(\Gamma_i|\Delta^*) \le c \operatorname{dist}(F_i(\Delta^*), \partial K)$$

for some constant c, which implies by rescaling that

$$\operatorname{Lip}(\Gamma_i'|\Delta^*) \le c \operatorname{dist}(F_i'(\Delta^*), \partial K_i').$$

By (15) and (16), the right hand side tends to c, so

$$\limsup_i \left(\operatorname{Lip}(\Gamma_i'|\Delta^*)\right) \le c. \tag{17}$$

If v is a vertex of Δ, then $\Gamma_i'(v)$ is a point in $\partial K_i'$ closest to $F_i'(v)$. Since since $F_i'(\cdot) \to 0$ and since $K_i' \to K'$ smoothly, this implies that

$$\limsup_{i\to\infty} \operatorname{dist}(\Gamma_i'(v), 0) = \operatorname{dist}(\partial K', 0) = 1. \tag{18}$$

By (17) and (18), the Γ_i' form an equicontinuous family, so after passing to a subsequence, we can assume that the Γ_i' converge uniformly to a Lipschitz map

$$\Gamma' : \partial\Delta \to \partial K'.$$

Now $\partial K'$ is smooth. Also, $k = \dim(\Delta) \le Q(K)$, so by definition of $Q(K)$, the $(k-1)$-dimensional homotopy of $\partial K'$ is trivial. Thus the map Γ' extends to a Lipschitz map $G' : \Delta \to \partial K'$.

By the smooth convergence $K_i' \to K'$ and by the bounded Lipschitz norm convergence $\Gamma_i' \to \Gamma'$, it follows that (for all sufficiently large i) there is a Lipschitz map

$$G_i' : \Delta \to \partial K_i'$$

such that G_i' extends Γ_i' and such that

$$\operatorname{Lip}(G_i') \le \operatorname{Lip}(G') + \delta_i \tag{19}$$

where $\delta_i \to 0$. We may assume that $G_i' : \partial\Delta \to K_i'$ is the extension of smallest Lipschitz norm. (This minimizing extension exists because $\partial K_i'$ is compact.) By passing to a subsequence, the G_i' converge uniformly to a limit map, which we may assume to be G'. (Otherwise redefine G' to be that limit map.)

In particular, the smooth convergence $\partial K_i' \to \partial K'$ implies that G_i' maps Δ to the regular part of $\partial K_i'$ (if i is sufficiently large).

Note that

$$\frac{\operatorname{Lip}(G_i)}{\operatorname{dist}(F_i(\Delta), \partial K)} = \frac{\operatorname{Lip}(G_i')}{\operatorname{dist}(F_i'(\Delta), \partial K_i')} = \frac{\operatorname{Lip}(G_i')}{1}$$

which is bounded as $i \to \infty$ by (19).

Similarly we have

$$\frac{\operatorname{diam}(F_i(\Delta) \cup G_i(\Delta))}{\operatorname{dist}(F_i(\Delta), \partial K)} = \frac{\operatorname{diam}(F_i'(\Delta) \cup G_i'(\Delta))}{\operatorname{dist}(F_i'(\Delta), \partial K_i')} = \frac{\operatorname{diam}(F_i'(\Delta) \cup G_i'(\Delta)}{1} \tag{20}$$

which converges to $\operatorname{diam}(\{0\} \cup G'(\Delta))$ as $i \to \infty$ (since $F_i' \to 0$ and $G_i' \to G'$ uniformly.) In particular, (20) is bounded as $i \to \infty$. □

5. Manifolds with Boundary

So far in this paper, the moving hypersurfaces $\partial K(t)$ under consideration have been hypersurfaces without boundary. Now we consider the case of hypersurfaces with boundary, the motion of the boundary being prescribed and the motion away from the boundary being by mean curvature flow (or possibly by other analogous flows.)

5.1. **Definition.** Let N be a smooth $(n+1)$-dimensional manifold-with-boundary. Let K be a closed subset of N. A point $p \in K$ is called a **regular point** of K provided

(1) p is an interior point of K, or
(2) $p \in N \setminus \partial N$ and N has a neighborhood U of p such that $K \cap U$ is diffeomorphic to a closed half-space in $\mathbf{R}^{n+1}$, or
(3) $p \in \partial N$ and N has a neighborhood U of p for which there is a diffeomorphism that maps U onto $\{x \in \mathbf{R}^{n+1} : x_1 \geq 0\}$ and that maps $K \cap U$ onto $\{x \in \mathbf{R}^{n+1} : x_1 \geq 0, x_2 \geq 0\}$.

Points in K that are not regular points are called singular points of K.

The following theorem should be thought of as a theorem about a moving hypersurface-with-boundary. At time t, the hypersurface is

$$M(t) = \partial K(t) = K(t) \cap \overline{N \setminus K(t)},$$

and its boundary is $\Gamma(t) := M(t) \cap \partial N$. In practice, the initial surface would be prescribed by prescribing $K(0)$, and the motion of the boundary would be prescribed by prescribing $\Gamma(t)$ or, equivalently, by prescribing $K(t) \cap N$. The geometric flow would then determine the moving region $K(t)$ or, equivalently, the moving hypersurface $M(t)$.

We first give an abstract version of the main theorem of this section (afterwards, in Theorem 5.3, we specialize to mean curvature flow):

5.2. **Theorem.** *Let $t \in [a, b] \mapsto K(t)$ be a one-parameter family of compact subsets of a smooth, $(n+1)$-dimensional Riemannian manifold-with-boundary N, and let*

$$M(t) = \partial K(t) = K(t) \cap \overline{N \setminus K(t)}.$$

Assume that there is a collared neighborhood $U \subset N$ of ∂N such that each $M(t) \cap U$ is a smooth, embedded manifold-with-boundary, the boundary being $M(t) \cap \partial N$, and that $M(t) \cap U$ depends smoothly on t for $t \in [a, b]$. Assume that $M(t) \cap U$ is never tangent to ∂N. Assume also that

(1) $K(t) \subset K(T)$ *for* $T \leq t$.
(2) $K(a) \cap K(b)^c = \cup_{a \leq t < b} M(t)$.
(3) $M(t) \cap M(T) \subset \partial N$ *for* $t \neq T$.

If

$$Q(K(t)) \geq m \text{ for all } t \in [a, b),$$

then the pair $(K(b)^c, K(a)^c)$ is m-connected.

Note that hypothesis (3) allows the boundary of $M(t)$ to be fixed or to move. Note also that the hypotheses imply that the singularities of $K(t)$ lie in the interior of N.

If the $M(t)$'s are disjoint, Theorem 5.2 can be proved exactly as Theorem 4.4 was proved. In the general case, we can reduce to the case of disjoint $M(t)$'s by replacing N by

$$N' := N \cap \{x : \operatorname{dist}(x, \partial N) \geq \delta\}$$

for some sufficiently small $\delta > 0$, and by replacing each $K(t)$ by $K(t) \cap N'$.

(For the proof, it is useful to note that if $F : (\mathbf{B}^k, \partial\mathbf{B}^k) \to (K(t)^c, K(a)^c)$, then F is homotopic in $(K(t)^c, K(a)^c)$ to a map G whose image lies in the interior of N. To see this, let $\mathbf{v}$ be any vectorfield on N that is equal on ∂N to the unit normal pointing into N. Now flow by that vectorfield for a short time to push F into the interior of N.)

In the case of mean curvature flow, we have the following theorem. In the statement of the the theorem, the boundary $\Gamma(t)$ of the moving surface $M(t) := \partial K(t)$ is given by giving a region $V(t)$ in ∂N such that $\Gamma(t) = \partial V(t)$.

5.3. **Theorem.** *Let N be a smooth, compact, connected $(n+1)$-dimensional Riemannian manifold with boundary with $n < 7$. Let $t \in [0, \infty) \mapsto V(t)$ be a smooth, one-parameter family of compact, smooth, n-dimensional manifolds with boundary in ∂N such that $V(t') \subset V(t)$ for $t \geq t'$. Let K be a closed subset of N such that ∂K is a smooth, compact, connected manifold-with-boundary such that $K \cap \partial N = V(0)$, and such that ∂K is smooth with mean curvature at each point a nonnegative multiple of the unit normal that points into K, and such that ∂K is nowhere tangent to ∂N.*

If ∂K is a minimal surface (i.e., has mean curvature 0 at all points), assume[4] also that $V(t) \neq V(0)$ for $t \neq 0$.

Let $t \in [0, \infty) \mapsto M(t)$ be the solution obtained by elliptic regularization of mean curvature flow such that $M(0) = \partial K$ and such that $\partial M(t) = \partial V(t)$ for all t.

Then each $M(t)$ is the boundary in N of a region $K(t) \subset K$. The singularities of the flow form a compact subset of the interior of N and all have convex type.

In particular, if the Gaussian densities of the singularities in the time interval $a \leq t < b$ are all $\leq d_m$, then $t \mapsto K(t)$ satisfies all the hypotheses of Theorem 5.2, and therefore the pair $(K(b)^c, K(a)^c)$ is m-connected.

If $n \geq 7$, the Theorem remains true provided the metric on N is flat and provided $M(t)$ is smooth for some $t \geq b$.

The theorem should be true for all n without the somewhat peculiar assumptions in the last sentence of the theorem. Those assumptions are needed only because without them we do not know how to prove that the singularities of the flow have convex type.

Proof. Except for the assertion that the pair $(K(b)^c, K(a)^c)$ is m-connected, this is proved in [Whi11]. The m-connectivity of $(K(b)^c, K(a)^c)$ then follows by Theorem 5.2 and Proposition 4.3. □

6. The topology of the moving regions

The theorems described so far are about the topology of the exteriors of the moving regions $K(t)$. Using standard duality theorems of topology, we can draw conclusions about the changing topology of the regions themselves.

[4] This assumption guarantees that the surface starts moving immediately.

6.1. **Proposition.** *Let X be a compact orientable $(n+1)$-dimensional manifold-with-boundary. Suppose*

$$\partial X = A \cup B$$

where A and B are compact n-manifolds-with-boundary, and that $A \cap B = \partial A = \partial B$. If (X, A) is m-connected, then

$$\mathrm{H}_k(X, B) = 0 \tag{21}$$

for $k > n - m$.

Proof. Let $p \leq m$. Since (X, A) is m-connected,

$$\mathrm{H}_p(X, A) = \mathrm{H}_{p-1}(X, A) = 0.$$

It follows that $\mathrm{H}^p(X, A) = 0$. (This is easy to prove directly, but it is also a special case of the Universal Coefficients Theorem [Hat02, Theorem 3.2, p. 195].) By the Poincare-Lefschetz Duality Theorem [Hat02, Theorem 3.43, p. 254],

$$\mathrm{H}^p(X, A) \cong \mathrm{H}_{n+1-p}(X, B).$$

Thus $\mathrm{H}_{n+1-p}(X, B) = 0$. This holds for every $p \leq m$, so (21) holds for every $k > n - m$. □

6.2. **Theorem.** *Suppose, in Theorems 3.1, 4.4, 5.2, or 5.3, that the surfaces $\partial K(a)$ and $\partial K(b)$ are smooth. Then:*

(1) $\mathrm{H}_k(K(a), K(b)) = 0$ *for all $k > n - m$.*
(2) *The map $\iota_\# : \mathrm{H}_k(K(b)) \to \mathrm{H}_k(K(a))$ is an isomorphism for $k > n - m$ and is an injection for $k = n - m$.*

Proof. Consider first the case of Theorems 3.1 and 4.4. Let $A = \partial K(a)$, $B = \partial K(b)$, and $X = K(a) \setminus \mathrm{interior}(K(b))$. By those theorems, $(K(b)^c, K(a)^c)$ is m-connected. Since $\partial K(a)$ and $\partial K(b)$ are smooth, m-connectivity of (X, A) follows easily. Thus by Proposition 6.1,

$$\mathrm{H}_k(X, B) = 0 \tag{22}$$

for $k > n - m$. But by excision, $\mathrm{H}_k(X, B) = \mathrm{H}_k(K(a), K(b))$. This proves assertion (1). Assertion (2) follows immediately by the long exact sequence for $\mathrm{H}_*(K(a), K(b))$.

Now consider the case of Theorems 5.2 and 4.4. For simplicity, let us suppose that the boundary of the moving surface is fixed:

$$\partial M(t) \equiv \Gamma \qquad (t \in [a, b]).$$

Then one applies Proposition 6.1 exactly as in the previous paragraph. In that paragraph, the A and B were disjoint, whereas now $A \cap B = \Gamma$. (Of course X has corners, so the boundary is not smooth, but X is a topological manifold with boundary, so Proposition 6.1 applies.)

If the boundary of the surface $M(t)$ is not fixed, one can still apply Proposition 6.1: one lets $A = \partial K(a)$ and

$$B = (\partial K(b)) \cup (\cup_{t \in [a,b]} \Gamma(t))$$

and argues as before to get (22). It follows that

$$\mathrm{H}_k(X, \partial K(b)) = 0$$

since (X, B) is homotopy equivalent to $(X, \partial K(b))$. Finally, one uses excision and the long exact sequence for $\mathrm{H}_*(K(a), K(b))$ exactly as before to get (1) and (2). □

6.3. **Theorem.** *Let $t \mapsto K(t)$ be as in Theorems 3.1 or 5.3, but without the assumption about Gaussian density of singularities.*

If there is an integral p-cycle in $K(a)^c$ that bounds a $(p+1)$-chain in in $K(b)^c$ but not in $K(a)^c$, then there is a singularity (in the time interval $a < t < b$) whose Gaussian density is $\geq d_p$.

If $\partial K(a)$ and $\partial K(b)$ are smooth and if there is an integral q-cycle in $K(b)$ that bounds in $K(a)$ but not in $K(b)$, then there is a singularity in the time interval $a < t < b$ with Gaussian density $\geq d_{n-q-1}$.

To illustrate Theorem 6.3, suppose $K(a)$ is connected but that $K(b)$ is not connected. Let x and y be points in $K(b)$ that lie in different connected components of $K(b)$. Then the 0-cycle $[x] - [y]$ (i.e, the cycle consisting of the point x with multiplicity 1 and the point y with multiplicity -1) bounds a 1-chain in $K(a)$ but not in $K(b)$. Thus according to Theorem 6.3 there must be a singularity with Gaussian density $\geq d_{n-1}$.

Proof. If R is an integral p-cycle in $K(a)^c$ that bounds a $(p+1)$-chain S in $K(b)^c$ but does not bound any chain in $K(a)^c$, then S represents a nonzero element of $\mathrm{H}_{p+1}(K(b)^c, K(a)^c)$, so $(K(b)^c, K(a)^c)$ is not $(p+1)$-connected, so (by Theorem 3.1 or 5.3), there must be a singularity (in the time interval $a < t < b$) whose Gaussian density is $> d_{p+1}$ and therefore $\geq d_p$.

Similarly, if R is an integral q-cycle in $K(b)$ that bounds a $(q+1)$-chain S in $K(a)$ but not in $K(b)$, then S represents an nonzero element of $\mathrm{H}_{q+1}(K(a), K(b))$, so by Theorem 6.2, there is a singularity with Gaussian density $\geq d_{n-q-1}$. $\square$

7. The Persistence of Neck-Pinches

7.1. **Theorem.** *Suppose $1 \leq m \leq n$. Let $\mathcal{F}$ be the family of all compact, mean convex regions in $\mathbf{R}^{n+1}$ with smooth boundary. Let $\mathcal{F}_m$ be the set of $K \in \mathcal{F}$ such that the mean curvature flow with initial surface ∂K has a shrinking $\mathbf{S}^m \times \mathbf{R}^{n-m}$ singularity. Then $\mathcal{F}_m$ has nonempty interior.*

Of course the meaning of "interior" depends on the choice of topology on $\mathcal{F}$. Here we use the topology in which in K_i converges to K if and only if ∂K_i converges to ∂K in C^1. (We could use C^k for any k with $1 \leq k \leq \infty$, but C^1 gives the strongest result.)

We remark that the degenerate neckpinches mentioned in the introduction show that $\mathcal{F}_m$ is not, in general, an open subset of $\mathcal{F}$.

Proof. Let $\mathcal{C}$ be the set of $K \in \mathcal{F}$ such that K^c has nontrivial m^{th} homotopy and such that the mean curvature flow with initial surface ∂K has no singularities of Gaussian density $\geq d_{m-1}$ (or, equivalently, such that each singularity of the flow has Gaussian density $< d_{m-1}$). Since the Gaussian density at a spacetime point (x, t) is upper semicontinuous as a function of the spacetime point and of the flow, the set $\mathcal{C}$ is an open subset of $\mathcal{F}$. By Corollary 1.2, the resulting mean curvature flow has an $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity for some $k \leq m$. But by definition of $\mathcal{C}$, we have $d_k < d_{m-1}$, which implies (see (1)) that $k > m-1$ and thus that $k = m$. Hence $\mathcal{C}$ is a subset of $\mathcal{F}_m$.

We have shown that $\mathcal{C}$ is open and that $\mathcal{C}$ is contained in $\mathcal{F}_m$. It remains only to show that $\mathcal{C}$ is nonempty. Let S be a round $(n-m)$-sphere in $\mathbf{R}^{n+1}$. For $\epsilon > 0$, let $K(\epsilon)$ be the set of points at distance $\leq \epsilon$ from S. If $\epsilon > 0$ is sufficiently small,

$K(\epsilon)$ will be mean convex. Fix such an ϵ. By symmetry, $K(\epsilon)$ collapses under mean curvature flow to a round $(n-m)$-sphere, from which it easily follows that the singularities are all $\mathbf{S}^m \times \mathbf{R}^{n-m}$ singularities. Also, $K(\epsilon)^c$ has nontrivial k^{th} homotopy, so $K(\epsilon) \in \mathcal{C}$ and therefore $\mathcal{C}$ is nonempty. $\square$

8. Appendix

Here we give a proof of the general position principle used in the proof of Theorem 4.4.

8.1. **Proposition.** *Let N be a smooth d-dimensional manifold without boundary and let S be a subset of N with Hausdorff $(d-k)$-dimensional measure 0. Then the collection $\mathcal{C}$ of smooth maps $F : \mathbf{B}^k \to N$ such that $F(\mathbf{B}^k) \cap S = \emptyset$ is dense in the set of all smooth maps from $\mathbf{B}^k$ to N.*

Proof. First consider the case $N = \mathbf{R}^d$. Let $F : \mathbf{B}^k \to N$ be a smooth map. We will prove the proposition in this case by showing

$$\text{If } F \in C^\infty(\mathbf{B}^k, \mathbf{R}^d), \text{ then } F(\cdot) + v \in \mathcal{C} \text{ for almost every } v \in \mathbf{R}^d. \tag{23}$$

The set $\Pi^{-1}(S) = \mathbf{B}^k \times S$ has $k+(d-k)$-dimensional (i.e., d-dimensional) measure 0. (Here $\Pi : \mathbf{B}^k \times \mathbf{R}^d \to \mathbf{R}^d$ is the projection map.) Therefore its diffeomorphic image $\phi(\Pi^{-1}(S))$ under the diffeomorphism

$$\phi : (x, y) \in \mathbf{B}^k \times \mathbf{R}^n \mapsto (x, y - F(x))$$

has d-dimensional measure 0. Hence the projected image $\Pi(\phi(\Pi^{-1}(S)))$ of $\phi(\Pi^{-1}(S))$ in $\mathbf{R}^d$ has Lebesgue measure 0:

$$\mathcal{L}^d(\Pi(\phi(\Pi^{-1}(S))) = 0. \tag{24}$$

Now

$$v \in \Pi(\phi(\Pi^{-1}(S))) \iff (x, v) \in \phi(\Pi^{-1}(S)) \text{ for some } x \in \mathbf{B}^k, \tag{25}$$

and

$$\begin{aligned} (x, v) \in \phi(\Pi^{-1}(S)) &\iff \phi^{-1}(x, v) \in \Pi^{-1}(S) \\ &\iff (x, F(x) + v) \in \Pi^{-1} S \\ &\iff F(x) + v \in S \end{aligned} \tag{26}$$

The desired conclusion (23) follows immediately from (24), (25), and (26). This completes the proof in the case $N = \mathbf{R}^d$.

For a general manifold N, we may assume that N is a smooth submanifold of some Euclidean space $\mathbf{R}^{d+j}$. Let U be an open subset of $\mathbf{R}^{d+j}$ that contains N and for which the nearest point retraction $\pi : U \to N$ exists and is smooth.

Let $F : \mathbf{B}^k \to N$ be a smooth map, and let $\delta = d(F(\mathbf{B}^k), U^c)$. Now the set $\pi^{-1}(S)$ has $(d-k)+j$ dimensional measure 0, or, equivalently $(d+j)-k$-dimensional measure 0. Thus by (23), the map $F(\cdot)+v$ has image disjoint from $\pi^{-1}(S)$ for almost every $v \in \mathbf{R}^{d+j}$ with $|v| < \delta$. Therefore the map $\pi(F(\cdot)+v)$ has image disjoint from S for almost every $v \in \mathbf{R}^{d+j}$ with $|v| < \delta$. $\square$

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